Complex manifolds with split tangent bundle

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Introduction

The theme of this note is to investigate when the tangent bundle of a compact complex manifold X splits as a direct sum of sub-bundles. This occurs typically when the universal covering space $\widetilde{\mathrm{X}}$ of X splits as a product $\prod\limits_{i\in\mathrm{I}}\mathrm{U}_i$ of manifolds on which the group $\pi_1(\mathrm{X})$ acts diagonally (that is, $\pi_1(\mathrm{X})$ acts on each U_i and its action on $\widetilde{\mathrm{X}}=\prod\mathrm{U}_i$ is the diagonal action $g.(u_i)=(gu_i)$): the vector bundles [2] $\mathrm{T}_{\mathrm{U}_i}$ on $\widetilde{\mathrm{X}}$ are stable under $\pi_1(\mathrm{X})$, hence the decomposition $\mathrm{T}_{\widetilde{\mathrm{X}}}=\bigoplus\limits_i \mathrm{T}_{\mathrm{U}_i}$ descends to a direct sum decomposition of T_{X}. For *Kähler* manifolds, it is tempting to conjecture that the converse is true, namely that any direct sum decomposition of the tangent bundle T_{X} (perhaps with the additional assumption that the direct summands are integrable) gives rise to a splitting of the universal covering. We will show that this is indeed the case in three different situations:

a) X admits a Kähler-Einstein metric;

b) T_{X} is a direct sum of line bundles of negative degree;

c) X is a Kähler surface.

Case *a*) is a direct consequence of the fact that on a compact Kähler-Einstein manifold, any endomorphism of the tangent bundle is parallel (this idea appears for instance in [Y], and in a more implicit form in [K]). Case *b*) is a slight improvement of a uniformization result of Simpson [S]. To treat case *c*) we use the classification of surfaces and some simple remarks about connections. The result in this case is actually an easy consequence of the paper [K-O], where the authors classify surfaces with a holomorphic conformal structure – this turns out to be closely related to the question we are studying here. However we found simpler and more enlightening to give an independent proof rather than extracting from [K-O] the pieces of information that we need.

In § 2 we give a few examples which show that for non-Kähler manifolds a splitting of the tangent bundle does not necessarily imply a splitting of the universal covering.

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[2] Throughout the paper we will abuse notation and write $\mathrm{T}_{\mathrm{U}_i}$ instead of $pr_i^*\mathrm{T}_{\mathrm{U}_i}$.

1. Kähler-Einstein manifolds

Theorem A .– *Let* X *be a compact complex manifold admitting a Kähler-Einstein metric. Assume that the tangent bundle of* X *has a decomposition* $\mathrm{T}_{\mathrm{X}} = \bigoplus_{i\in \mathrm{I}} \mathrm{E}_i$. *Then the universal covering space of* X *is a product* $\prod_{i\in\mathrm{I}} \mathrm{U}_i$ *of complex manifolds, in such a way that the decomposition* $\mathrm{T}_{\mathrm{X}} = \bigoplus_{i\in \mathrm{I}} \mathrm{E}_i$ *lifts to the decomposition* $\mathrm{T}_{\Pi \mathrm{U}_i} = \bigoplus_{i\in\mathrm{I}} \mathrm{T}_{\mathrm{U}_i}$ *; the group* $\pi_1(\mathrm{X})$ *acts diagonally on* $\prod_{i\in\mathrm{I}} \mathrm{U}_i$.

The proof follows closely that of thm. 2.1 in [Y] (I am indebted to J. Wahl for pointing out this reference).

Proof: (1.1) As a consequence of the Bochner formula, every endomorphism of T_{X} is parallel [K]. This applies in particular to the projectors associated to the direct sum decomposition of T_{M} ; therefore the sub-bundles E_i are preserved by the hermitian connection, hence the holonomy representation of X is the direct sum of a family of representations corresponding to the E_i 's. By the De Rham theorem, the universal covering space of X splits as a product $\prod_{i\in\mathrm{I}} \mathrm{U}_i$, such that the decomposition $\mathrm{T}_{\mathrm{X}} = \bigoplus_{i\in \mathrm{I}} \mathrm{E}_i$ pulls back to the decomposition $\mathrm{T}_{\Pi \mathrm{U}_i} = \bigoplus_{i\in\mathrm{I}} \mathrm{T}_{\mathrm{U}_i}$.

(1.2) The last assertion follows from the following simple observation: *if a group* Γ *acting on a product* $\prod_{i\in\mathrm{I}} \mathrm{U}_i$ *preserves the decomposition* $\mathrm{T}_{\Pi \mathrm{U}_i} = \bigoplus_{i\in\mathrm{I}} \mathrm{T}_{\mathrm{U}_i}$ *, it acts diagonally.* Let indeed γ be an automorphism of $\prod \mathrm{U}_i$; for $j \in \mathrm{I}$, put $\gamma_j = pr_j \circ \gamma$. The condition $\gamma^* \mathrm{T}_{\mathrm{U}_j} = \mathrm{T}_{\mathrm{U}_j}$ means that the partial derivatives of γ_j in the directions of U_k for $k \neq j$ vanish, hence $\gamma_j((u_i)_{i\in\mathrm{I}})$ depends only on u_j , which gives our claim. ∎

2. Non-Kähler examples

In this section we give examples of manifolds for which the tangent bundle is a direct sum of line bundles, but which do not satisfy the conclusions of Theorem A.

(2.1) *Hopf manifolds*

Let $\mathrm{T} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ be a diagonal matrix, with $0 < |\alpha_i| < 1$ for each i . The cyclic group $\mathrm{T}^{\mathbf{Z}}$ generated by T acts freely and properly on $\mathbf{C}^n - \{0\}$; the quotient X is a compact complex manifold, called a Hopf manifold. For each non-zero complex number θ , denote by L_θ the flat line bundle associated to the character of $\pi_1(\mathrm{X}) = \mathrm{T}^{\mathbf{Z}}$ mapping T to θ ; in other words, L_θ is the quotient of the trivial line bundle $(\mathbf{C}^n - \{0\}) \times \mathbf{C}$ by the action of the automorphism (T, θ) . By construction we have $\mathrm{T}_{\mathrm{X}} = \bigoplus_{i=1}^{n} \mathrm{L}_{\alpha_i}$, but the universal covering space $\mathbf{C}^n - \{0\}$ of X is clearly not a product.

(2.2) *Complex compact nilmanifolds*

These are compact manifolds $\mathrm{X} = \mathrm{G}/\Gamma$, where G is a nilpotent complex Lie

group and Γ a discrete subgroup of G. We may assume that G is simply-connected and non-commutative (to exclude the trivial case of complex tori). A well-known example is the Iwasawa manifold $\mathrm{U}(\mathbf{C})/\mathrm{U}(\mathbf{Z}[i])$, where U is the group of upper-triangular 3×3 matrices with diagonal entries 1; many examples can be obtained in an analogous way.

The tangent bundle of $\mathrm{X} = \mathrm{G}/\Gamma$ is trivial, and its universal covering space G is isomorphic to $\mathbf{C}^n$; however we claim that whatever isomorphism $\mathrm{G} \xrightarrow{\sim} \mathbf{C}^n$ we choose, *the action of* Γ *cannot be diagonal*. Indeed if Γ acts diagonally, the standard trivialization of $\mathrm{T}_{\mathbf{C}^n}$ deduced from the coordinate system descends to a trivialization of T_{X}. Any such trivialization lifts to a trivialization of T_{G} defined by a basis of right invariant vector fields; therefore the standard trivialization of $\mathrm{T}_{\mathbf{C}^n}$ is G-equivariant. In view of 1.2 this means that G itself acts diagonally on $\mathbf{C}^n$, hence G embeds into $\mathrm{Aut}(\mathbf{C})^n$. Now any nilpotent connected subgroup of the affine group $\mathrm{Aut}(\mathbf{C})$ is commutative, so we conclude that G is commutative, contrary to our hypothesis.

3. Simpson's uniformization result

The following lemma, which is a variation on the Baum-Bott theorem [B-B], will allow us to slightly improve Simpson's result:

Lemma 3.1 .– *Let* X *be a complex manifold, and* E *a direct summand of* T_{X}. *The Atiyah class* $\mathrm{at}(\mathrm{E}) \in \mathrm{H}^1(\mathrm{X}, \Omega^1_{\mathrm{X}} \otimes \mathcal{E}nd(\mathrm{E}))$ *comes from* $\mathrm{H}^1(\mathrm{X}, \mathrm{E}^* \otimes \mathcal{E}nd(\mathrm{E}))$. *In particular, any class in* $\mathrm{H}^r(\mathrm{X}, \Omega^r_{\mathrm{X}})$ *given by a polynomial in the Chern classes of* E *vanishes for* $r > \mathrm{rk}(\mathrm{E})$.

Proof: Write $\mathrm{T}_{\mathrm{X}} = \mathrm{E} \oplus \mathrm{F}$; let $p : \mathrm{T}_{\mathrm{X}} \to \mathrm{E}$ be the corresponding projection. For any sections U of E and V of F over some open subset of X, put $\mathrm{D}_{\mathrm{V}}\mathrm{U} = p([\mathrm{V}, \mathrm{U}])$. This expression is $\mathcal{O}_{\mathrm{X}}$-linear in V and satisfies the Leibnitz rule $\mathrm{D}_{\mathrm{V}}(f\mathrm{U}) = f\mathrm{D}_{\mathrm{V}}(\mathrm{U}) + (\mathrm{V}f)\mathrm{U}$, so that D is a F-*connection* on E [B-B]: if we denote by $\mathcal{D}^1(\mathrm{E})$ the sheaf of differential operators $\Delta : \mathrm{E} \to \mathrm{E}$, of degree ≤ 1, whose symbol $\sigma(\Delta)$ is scalar, this means that D defines an $\mathcal{O}_{\mathrm{X}}$-linear map $\mathrm{F} \to \mathcal{D}^1(\mathrm{E})$ such that $\sigma(\mathrm{D}_{\mathrm{V}}) = \mathrm{V}$ for all local sections V of F. Thus the exact sequence

$$0 \to \mathcal{E}nd(\mathrm{E}) \longrightarrow \mathcal{D}^1(\mathrm{E}) \xrightarrow{\sigma} \mathrm{T}_{\mathrm{X}} \to 0$$

splits over the sub-bundle $\mathrm{F} \subset \mathrm{T}_{\mathrm{X}}$; therefore its extension class $\mathrm{at}(\mathrm{E}) \in \mathrm{H}^1(\mathrm{X}, \Omega^1_{\mathrm{X}} \otimes \mathcal{E}nd(\mathrm{E}))$ vanishes in $\mathrm{H}^1(\mathrm{X}, \mathrm{F}^* \otimes \mathcal{E}nd(\mathrm{E}))$, hence comes from $\mathrm{H}^1(\mathrm{X}, \mathrm{E}^* \otimes \mathcal{E}nd(\mathrm{E}))$. The last assertion follows from the definition of the Chern classes in terms of the Atiyah class. ∎

We denote as usual by $\mathbf{H}$ the Poincaré upper half-space.

Theorem B .– *Let* X *be a compact Kähler manifold, with a Kähler class* ω *. Assume that the tangent bundle* $\mathrm{T_X}$ *is a direct sum of line bundles* $\mathrm{L}_1,\ldots,\mathrm{L}_n$ *with* $\omega^{n-1}.\,c_1(\mathrm{L}_i)<0$ *for each* i *. Then the universal covering space of* X *is* $\mathbf{H}^n$ *, and the decomposition* $\mathrm{T_X}=\oplus\mathrm{L}_i$ *lifts to the canonical decomposition* $\mathrm{T}_{\mathbf{H}^n}=(\mathrm{T}_{\mathbf{H}})^{\oplus n}$ *.*

Proof: This is Cor. 9.7 of [S], except that Simpson makes the extra hypothesis $\omega^{n-2}.\,(c_1(\mathrm{X})^2-2c_2(\mathrm{X}))=0$ (the assertion about the compatibility of decompositions is not stated in *loc. cit.*, but follows directly from the proof). Now lemma 3.1 gives $c_1(\mathrm{L}_i)^2=0$ for each i, hence $c_1(\mathrm{X})^2-2c_2(\mathrm{X})=0$. ∎

4. The surface case

Theorem C .– *Let* X *be a compact complex surface. The tangent bundle of* X *splits as a direct sum of two line bundles if and only if one of the following occurs:*

a) *The universal covering space of* X *is a product* $\mathrm{U}\times\mathrm{V}$ *of two (simply-connected) Riemann surfaces and the group* $\pi_1(\mathrm{X})$ *acts diagonally on* $\mathrm{U}\times\mathrm{V}$ *; in that case the given splitting of* $\mathrm{T_X}$ *lifts to the direct sum decomposition* $\mathrm{T}_{\mathrm{U}\times\mathrm{V}}=\mathrm{T_U}\oplus\mathrm{T_V}$ *.*

b) X *is a Hopf surface, with universal covering space* $\mathbf{C}^2-\{0\}$ *. Its fundamental group is isomorphic to* $\mathbf{Z}\oplus\mathbf{Z}/m\mathbf{Z}$ *, for some integer* $m\geq 1$ *; it is generated by a diagonal automorphism* $(x,y)\mapsto(\alpha x,\beta y)$ *with* $|\alpha|\leq|\beta|<1$ *, and a diagonal automorphism* $(x,y)\mapsto(\lambda x,\mu y)$ *where* λ *and* μ *are primitive* m*-th roots of* 1 *.*

As a corollary, for Kähler surfaces we see that any direct sum decomposition of the tangent bundle gives rise to a splitting of the universal covering, as announced in the introduction.

(4.1) Before starting the proof we will need a few preliminaries. From now on we denote by X a compact complex surface; we assume given a direct sum decomposition $\Omega^1_{\mathrm{X}}\cong\mathrm{L}\oplus\mathrm{M}$. By lemma 3.1 (or by [B-B]) the Chern class $c_1(\mathrm{L})\in\mathrm{H}^1(\mathrm{X},\Omega^1_{\mathrm{X}})$ belongs to the subspace $\mathrm{H}^1(\mathrm{X},\mathrm{L})$, and similarly for M. As a consequence, we get:

(4.2) We have $\mathrm{L}^2=\mathrm{M}^2=0$, and therefore $c_1^2(\mathrm{X})=2\,\mathrm{L}.\mathrm{M}=2c_2(\mathrm{X})$.

The following consequence is less obvious.

Proposition 4.3 .– *Let* C *be a smooth rational curve in* X *. Then* $\mathrm{C}^2\geq 0$ *.*

Proof: Put $\mathrm{C}^2=-d$ and assume $d>0$. Since $\mathrm{H}^1(\mathrm{C},\mathcal{O}_\mathrm{C}(d+2))=0$, the exact sequence

$$0\to\mathcal{O}_\mathrm{C}(d)\longrightarrow\Omega^1_{\mathrm{X}\,|\mathrm{C}}\longrightarrow\Omega^1_\mathrm{C}\to 0$$

splits, giving an isomorphism $\Omega^1_{\mathrm{X}\,|\mathrm{C}}\cong\mathcal{O}_\mathrm{C}(d)\oplus\mathcal{O}_\mathrm{C}(-2)$. Thus one of the line bundles L or M, say L, satisfies $\mathrm{L}_{|\mathrm{C}}\cong\mathcal{O}_\mathrm{C}(d)$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^1(\mathrm{X},\mathrm{L}) & \longrightarrow & \mathrm{H}^1(\mathrm{X},\Omega^1_{\mathrm{X}}) \\
\downarrow & & \downarrow \\
\mathrm{H}^1(\mathrm{C},\mathrm{L}_{|\mathrm{C}}) & \longrightarrow & \mathrm{H}^1(\mathrm{C},\Omega^1_{\mathrm{C}})
\end{array} \quad ;$$

since $d>0$ we have $\mathrm{H}^1(\mathrm{C},\mathrm{L}_{|\mathrm{C}})=0$; thus $c_1(\mathrm{L})$ goes to 0 in $\mathrm{H}^1(\mathrm{C},\Omega^1_{\mathrm{C}})$, which means $d=0$, a contradiction. ∎

(4.4) We shall come across situations where the vector bundle $\Omega^1_{\mathrm{X}}=\mathrm{L}\oplus\mathrm{M}$ appears as an extension

$$0\to\mathrm{P}\longrightarrow\Omega^1_{\mathrm{X}}\xrightarrow{\ p\ }\mathrm{Q}\to 0$$

of two line bundles P and Q. In that case,

– either the restriction of p to one of the direct summands of Ω^1_{X}, say M, is surjective; then the exact sequence splits, Q is isomorphic to M and P to L;

– or the restriction of p to both L and M is not surjective; then there exists effective (non-zero) divisors A and B, whose supports do not intersect, such that $\mathrm{L}\cong\mathrm{Q}(-\mathrm{A})$, $\mathrm{M}\cong\mathrm{Q}(-\mathrm{B})$ and $\mathrm{P}\cong\mathrm{Q}(-\mathrm{A}-\mathrm{B})$; the exact sequence does *not* split.

In particular, if $\mathrm{Hom}(\mathrm{P},\mathrm{Q})=0$, the exact sequence splits.

(4.5) Finally we will need some classical facts about connections (see [E]). Let $p:\mathrm{M}\to\mathrm{B}$ be a smooth holomorphic map between complex manifolds, whose fibres are isomorphic to a fixed variety F. A *connection* on p is a splitting of the exact sequence

$$0\to p^*\Omega^1_{\mathrm{B}}\longrightarrow\Omega^1_{\mathrm{M}}\longrightarrow\Omega^1_{\mathrm{M}/\mathrm{B}}\to 0\ ,$$

that is a sub-bundle $\mathrm{L}\subset\Omega^1_{\mathrm{M}}$ mapping isomorphically onto $\Omega^1_{\mathrm{M}/\mathrm{B}}$; the connection is flat (or integrable) if $d\mathrm{L}\subset\mathrm{L}\wedge\Omega^1_{\mathrm{M}}$ (this is automatic if B is a curve). In that case the group $\pi_1(\mathrm{B})$ acts on F by complex automorphisms, and M is the fibre bundle on B with fibre F associated to the universal covering $\widetilde{\mathrm{B}}\to\mathrm{B}$, that is the quotient of $\widetilde{\mathrm{B}}\times\mathrm{F}$ by the group $\pi_1(\mathrm{B})$ acting diagonally; the splitting $\Omega^1_{\mathrm{M}}=p^*\Omega^1_{\mathrm{B}}\oplus\mathrm{L}$ pulls back to the decomposition $\Omega^1_{\widetilde{\mathrm{B}}\times\mathrm{F}}=\Omega^1_{\widetilde{\mathrm{B}}}\oplus\Omega^1_{\mathrm{F}}$.

5. Proof of theorem C

(5.1) *Kodaira dimension* 2

If $\kappa(\mathrm{X})=2$, the canonical bundle K_{X} is ample by Prop. 4.3. The Aubin-Calabi-Yau theorem implies that X admits a Kähler-Einstein metric; we can therefore apply Theorem A.

(5.2) *Kodaira dimension* 1

If $\kappa(\mathrm{X})=1$, X admits an elliptic fibration $p:\mathrm{X}\to\mathrm{B}$. By 4.2 we have $c_2(\mathrm{X})=0$; this implies that the only singular fibres of p are multiples of smooth elliptic curves (see [B1], VI.4 and VI.5). For $b\in\mathrm{B}$, we write $p^*[b]=m(b)\,\mathrm{F}_b$, where F_b is a smooth elliptic curve; we have $m(b)\geq 1$ and $m(b)=1$ except for finitely many points. Put $\Delta=\sum_b(m(b)-1)\,\mathrm{F}_b$. We have an exact sequence

$$0\to p^*\Omega^1_{\mathrm{B}}(\Delta)\longrightarrow\Omega^1_{\mathrm{X}}\longrightarrow\omega_{\mathrm{X/B}}\to 0\ , \tag{5.3}$$

where $\omega_{\mathrm{X/B}}$ is the relative dualizing line bundle. Since $\chi(\mathcal{O}_{\mathrm{X}})=0$ by Riemann-Roch, we deduce from [B-P-V], V.12.2 and III.18.2, that $\omega_{\mathrm{X/B}}$ is a torsion line bundle. Since $\mathrm{K_X}=p^*\Omega^1_{\mathrm{B}}(\Delta)\otimes\omega_{\mathrm{X/B}}$, the hypothesis $\kappa(\mathrm{X})=1$ implies $\mathrm{Hom}(p^*\Omega^1_{\mathrm{B}}(\Delta),\omega_{\mathrm{X/B}})=0$, hence the exact sequence (5.3) splits by 4.4.

Let $\rho:\widetilde{\mathrm{B}}\to\mathrm{B}$ be the orbifold universal covering of (B,m): this is a ramified Galois covering, with $\widetilde{\mathrm{B}}$ simply-connected, such that the stabilizer of a point $\tilde{b}\in\widetilde{\mathrm{B}}$ is a cyclic group of order $m(\rho(\tilde{b}))$ (see for instance [K-O], lemma 6.1; note that because of the hypothesis $\kappa(\mathrm{X})=1$ and the formula for $\mathrm{K_X}$, there are at least 3 multiple fibers if B is of genus 0). Let $\widetilde{\mathrm{X}}$ be the normalization of $\mathrm{X}\times_{\mathrm{B}}\widetilde{\mathrm{B}}$. We have a commutative diagram

$$\begin{array}{ccc}\widetilde{\mathrm{X}} & \xrightarrow{\ \pi\ } & \mathrm{X}\\ {\scriptstyle\tilde{p}}\big\downarrow & & \big\downarrow{\scriptstyle p}\\ \widetilde{\mathrm{B}} & \xrightarrow{\ \rho\ } & \mathrm{B}\end{array}$$

where $\tilde{p}$ is smooth and π is étale ([B1], VI.7$'$). The exact sequence

$$0\to\tilde{p}^*\Omega^1_{\widetilde{\mathrm{B}}}\longrightarrow\Omega^1_{\widetilde{\mathrm{X}}}\longrightarrow\Omega^1_{\widetilde{\mathrm{X}}/\widetilde{\mathrm{B}}}\to 0$$

coincides with the pull back under π of the exact sequence (5.3); therefore p admits an integrable connection, given by the subbundle $\pi^*\mathrm{M}$ of $\Omega^1_{\widetilde{\mathrm{X}}}$. The result follows from 4.5 and 1.2.

(5.4) *Kodaira dimension* 0

Assume $\kappa(\mathrm{X})=0$. By 4.2 and the classification of surfaces, X is either a complex torus, a bielliptic surface, or a Kodaira surface. Complex tori and bielliptic surfaces fall into case *a*) of the theorem (a bielliptic surface is the quotient of a product $\mathrm{E}\times\mathrm{F}$ of elliptic curves by a finite abelian group acting diagonally).

A primary Kodaira surface has trivial canonical bundle and admits a smooth elliptic fibration $p:\mathrm{X}\to\mathrm{B}$. Thus the exact sequence (4.2) realizes Ω^1_{X} as an

extension of $\mathcal{O}_{\mathrm{X}}$ by $\mathcal{O}_{\mathrm{X}}$. Since $h^{1,0}(\mathrm{X})=1$, this extension is non-trivial, and it follows from 4.4 that Ω^1_{X} does not split.

A secondary Kodaira surface admits a primary Kodaira surface as a finite étale cover, hence its tangent bundle cannot split either.

(5.5) *Ruled surfaces*

We consider the case when X is algebraic and $\kappa(\mathrm{X})=-\infty$. By 4.2 and 4.3, X is a geometrically ruled surface, that is a projective bundle $p:\mathrm{X}\to\mathrm{B}$ over a curve. We again consider the exact sequence

$$0\to p^*\Omega^1_{\mathrm{B}}\longrightarrow\Omega^1_{\mathrm{X}}\longrightarrow\Omega^1_{\mathrm{X/B}}\to 0\ ;$$

since $\Omega^1_{\mathrm{X/B}}$ has negative degree on the fibres, we have $\mathrm{Hom}(p^*\Omega^1_{\mathrm{B}},\Omega^1_{\mathrm{X/B}})=0$, hence by 4.4 the above exact sequence splits: one of the direct summands of Ω^1_{X} defines an integrable connection for p. The result follows then from 4.5.

(5.6) *Inoue surfaces*

We now assume that X is not algebraic and $\kappa(\mathrm{X})=-\infty$, so that X is what is usually called a surface of type VII_0. These surfaces have $b_1=h^{0,1}=1$ and therefore $c_1^2+c_2=12\chi(\mathcal{O}_{\mathrm{X}})=0$; in our case this gives $c_1^2=c_2=0$ in view of 4.2, and finally $b_2=0$. Moreover we have $\mathrm{H}^0(\mathrm{X},\Omega^1_{\mathrm{X}}\otimes\mathrm{L}^{-1})\neq 0$. The surfaces with these properties have been completely classified by Inoue [I]: they are either Hopf surfaces, or belong to three classes of surfaces constructed by Inoue (*loc. cit.*).

We first consider the Inoue surfaces. The surfaces S_{M} of the first class are quotients of $\mathbf{H}\times\mathbf{C}$ by a group acting diagonally, hence they fall into case *a*) of the theorem.

The surfaces $\mathrm{S}^{(+)}_{\mathrm{N},p,q,r;t}$ of the second class are quotients of $\mathbf{H}\times\mathbf{C}$ by a group which does *not* act diagonally. This action leaves invariant the vector field $\partial/\partial z$ on $\mathbf{C}$, which therefore descends to a non-vanishing vector field v on X. This gives rise to an exact sequence

$$0\to\mathrm{K}_{\mathrm{X}}\xrightarrow{i(v)}\Omega^1_{\mathrm{X}}\xrightarrow{i(v)}\mathcal{O}_{\mathrm{X}}\to 0\ ,$$

which does not split since $h^{1,0}(\mathrm{X})=0$. We have $\mathrm{H}^0(\mathrm{X},\mathrm{K}_{\mathrm{X}}^{-1})=0$, for instance because X contains no curves; we infer from 4.4 that Ω^1_{X} does not split.

The surfaces $\mathrm{S}^{(-)}_{\mathrm{N},p,q,r}$ of the third class are quotients of certain surfaces of the second class by a fixed point free involution; therefore their tangent bundle does not split either.

(5.7) *Primary Hopf surfaces*

It remains to consider the class of Hopf surfaces, which are by definition the surfaces of class VII_0 whose universal covering space is $\mathbf{W}:=\mathbf{C}^2-\{0\}$. We

consider first the *primary* Hopf surfaces, which are quotients of $\mathbf{W}$ by the infinite cyclic group generated by an automorphism T of $\mathbf{W}$. According to [Ko], § 10, there are two cases to consider:

a) $\mathrm{T}(x,y)=(\alpha x,\beta y)$ for some complex numbers α,β with $0<|\alpha|\le|\beta|<1$;

b) $\mathrm{T}(x,y)=(\alpha^m x+\lambda y^m,\alpha y)$ for some positive integer m and non-zero complex numbers α,λ with $|\alpha|<1$.

As in 2.1, we denote by L_θ, for $\theta\in\mathbf{C}$, the flat line bundle associated to the character of $\pi_1(\mathrm{X})$ mapping T to θ. In case *a)* we find $\Omega^1_{\mathrm{X}}=\mathrm{L}_\alpha^{-1}\oplus\mathrm{L}_\beta^{-1}$, so the tangent bundle splits.

Let us consider case *b)*. The form dy on $\mathbf{W}$ satisfies $\mathrm{T}^*dy=\alpha\, dy$, hence descends to a form $\overline{dy}$ in $\mathrm{H}^0(\mathrm{X},\Omega^1_{\mathrm{X}}\otimes\mathrm{L}_\alpha)$; similarly the function y descends to a non-zero section of L_α. We have an exact sequence

$$0\to\mathrm{L}_\alpha^{-1}\xrightarrow{\overline{dy}}\Omega^1_{\mathrm{X}}\longrightarrow\mathrm{L}_\alpha^{-m}\to 0\ .$$

Since L_α has a non-zero section, the space $\mathrm{Hom}(\mathrm{L}_\alpha^{-1},\mathrm{L}_\alpha^{-m})$ is zero for $m>1$. Hence if Ω^1_{X} splits, we deduce from 4.4 that the exact sequence splits. This means that there exists a form $\overline{\omega}\in\mathrm{H}^0(\mathrm{X},\Omega^1_{\mathrm{X}}\otimes\mathrm{L}_\alpha^m)$ such that $\overline{\omega}\wedge\overline{dy}\neq 0$. Then $\overline{\omega}\wedge\overline{dy}$ is a generator of the trivial line bundle $\mathrm{K}_{\mathrm{X}}\otimes\mathrm{L}_\alpha^{m+1}$, hence pulls back to $c\,dx\wedge dy$ on $\mathbf{W}$, for some constant $c\neq 0$. Therefore the pull back ω of $\overline{\omega}$ to $\mathbf{W}$ is of the form $c\,dx+f(x,y)dy$ for some holomorphic function f on $\mathbf{C}^2$. The flat line bundle L_α^m carries a flat holomorphic connection ∇; the 2-form $\nabla\overline{\omega}$, which is a global section of $\mathrm{K}_{\mathrm{X}}\otimes\mathrm{L}_\alpha^m\cong\mathrm{L}_\alpha^{-1}$, is zero. This implies $d\omega=0$, so the function $f(x,y)$ is independent of x; let us write it $f(y)$. Now the condition $\mathrm{T}^*\omega=\alpha^m\omega$ reads $\alpha f(\alpha y)+c\lambda m y^{m-1}=\alpha^m f(y)$. Differentiating m times we find $f^{(m)}=0$, then differentiating $m-1$ times leads to a contradiction.

(5.8) *Secondary Hopf surfaces*

A secondary Hopf surface X is the quotient of $\mathbf{W}$ by a group Γ acting freely, containing a central, finite index subgroup generated by an automorphism T of the above type. We assume that Ω^1_{X} splits. The primary Hopf surface $\mathrm{Y}=\mathbf{W}/\mathrm{T}^{\mathbf{Z}}$ is a finite étale cover of X, so Ω^1_{Y} also splits; it follows from 5.7 that T is of type *a)*, and that Γ does not contain any transformation of type *b)*. According to [Ka], § 3, this implies that after an appropriate change of coordinates, the group Γ acts *linearly* on $\mathbf{C}^2$.

We claim that Γ is contained in a maximal torus of $\mathbf{GL}(2,\mathbf{C})$. This is clear if $\alpha\neq\beta$, because T is central in Γ. If $\alpha=\beta$, the direct sum decomposition of Ω^1_{X} pulls back to a decomposition $\Omega^1_{\mathrm{Y}}=\mathrm{L}_\alpha^{-1}\oplus\mathrm{L}_\alpha^{-1}$ (5.7), which for an appropriate choice of coordinates comes from the decomposition $\Omega^1_{\mathbf{W}}=\mathcal{O}_{\mathbf{W}}dx\oplus\mathcal{O}_{\mathbf{W}}dy$. Since Γ must preserve this decomposition, it is contained in the diagonal torus.

Thus we may identify Γ with a subgroup of $(\mathbf{C}^*)^2$; since it acts freely on $\mathbf{W}$, the first projection $\Gamma \to \mathbf{C}^*$ is injective. Therefore the torsion subgroup of Γ is cyclic, and we are in case *b*) of the theorem. ∎

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